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Dirichlet series with periodic coefficients

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Summary

In this paper we study periodic arithmetic functions $f = f(n)$ and the associated Dirichlet series $L(s,f)$ as a generalization of Dirichlet characters χ and the associated Dirichlet L - functions $L(s,\chi)$. Our results cover most of the results obtained so far by various authors, mostly for Dirichlet characters. In § 1 we give explicit evaluation of thevalues $L(k,f)$, and $L^{(k)}(1,f)$ $(k \in \mathbf{N})$. In § 2, we consider the product of several L -functions, more specifically, in § 2-1, we make an explicit determination of coefficients of the main term in the Piltz type divisor problem and related constants, in § 2-2, as a generalization of the Müller-Carlitz-Ayoub-Chowla-Redmond-Berndt theorem, give a Bessel series expression for the associated summatory functions and explicit determination of coefficients in the main term, and finally in § 2-3 we refer to the general product of L-series. Finally, in § 3 we give Chowla-Selberg type formulas in special cases.

§ 0. Preliminaries

Let $q \in \mathbf{N}$ be fixed throughout, and let $C(q)$ denote the $\mathbf{C}$ - vector space of all arithmetic functions $f : \mathbf{N} \to \mathbf{C}$ with period q, i.e. $f(n+q)=f(n)$, $^{\forall} n \in \mathbf{N}$. It is convinient to extend the domain of definition of f by letting for $^{\forall} n \in \mathbf{Z}$

$$f(n) = f(a), \quad n \equiv a \pmod{q}, \ 1 \le a \le q.$$

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Then, as noted by Yamamoto[11], $C(q)$ is an inner product space of dimension q, the inner product of $f,g \in C(q)$ being given by

$$(f,g) = \sum_{a \bmod q} f(a)\overline{g(a)}.$$

With respect to this , $\frac{1}{\sqrt{q}}\xi_a$, wih a running through a complete set of residues modulo q, and $\xi_a(n) = \exp(2\pi ian/q)$, additive characters of $\mathbf{Z}/q\mathbf{Z}$, form an orthonormal basis(abbriviated as ONB hereafter).

Yamamoto gives another basis of interest, which we will study at another ocassion ; in this paper we will be mainly concerned with $\left\{\frac{1}{\sqrt{q}}\xi_a \mid a \bmod q\right\}$.

Isomorphic with $C(q)$ is the vector space $D(q)$ of Dirichlet series associated to each $f \in C(q)$, namely

$$L(s,f) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

where $s = \sigma + it$ is the complex variable and the series is absolutely convergent, say for $\sigma > 1$(which is the case if $f(n) = O(n^{\varepsilon})$ for every $\varepsilon > 0$). Using the Hurwitz zeta function defined for $\alpha > 0$ by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s},$$

absolutely convergnt for $\sigma > 1$, we can find a basis of

$$D(q):\ q^{-s}\zeta(s,\{a/q\} + [1-\{a/q\}]),\ a \bmod q$$

where, $[\cdot]$ denotes the integral part of . and $\{\cdot\} = \cdot - [\cdot]$ the fractional part of . .

Thus

$$L(s,f) = q^{-s}\sum_{a \bmod q} f(a)\zeta(s,\{a/q\} + [1-\{a/q\}]),$$

which implies from the known fact about $\zeta(s,\alpha)$ that $L(s,f)$ is extended analytically over the whole plane and is holomorphic except possibly at $s=1$. Note that $L(s,f)$ is holomrphic at $s=1$ if and only if $f(1)+\cdots\cdots f(q)=0$.

We define the Fourier transform $\hat{f}$ of $f\in C(q)$ as the Fourier coefficients of f w.r.t. the basis $\left\{\frac{1}{\sqrt{q}}\xi_a\right\}$:

$$\hat{f}(n)=(f,\frac{1}{\sqrt{q}}\xi_n)=\frac{1}{\sqrt{q}}\sum_{a\bmod q} f(a)\exp(-2\pi i\frac{a}{q}n).$$

Corresponding to the Fourier series

$$f(n)=\sum_{a\bmod q}\hat{f}(a)\frac{1}{\sqrt{q}}\xi_a(n),$$

we have the representation of $L(s,f)$:

(0.1)
$$L(s,f)=\frac{1}{\sqrt{q}}\sum_{a\bmod q}\hat{f}(a)F(s,\frac{a}{q}),$$

where

$$F(s,\frac{a}{q})=\sum_{n=1}^{\infty}\frac{\xi_a(n)}{n^s}$$

is often referred to as the polylogarithm $l_s(a/q)$ with complex exponential argument and other times as the Lerch zeta-function, and is absolutely convergent for $\sigma>1$, in the first place. If $a\equiv 0\,(\bmod\, q)$, it reduces to the well-known Riemann zeta-function $\zeta(s)=\zeta(s,1)$, meromorphic with a simple pole at $s=1$ with residue 1(see(1.7)), while if $a\not\equiv 0\,(\bmod\, q)$, it can be extended to an integral function, say, by Lemma 2 of Milnor[8].

We note, in passing, the inversion formula(Funakura[5], Lemma 2) :An arithmeticai function has period q if and only if $\hat{\hat{f}}(n) = f(-n)$ holds for all n.We define the even part (f_{even}) and odd part (f_{odd}) of f by

$$f_{even}(n) = \frac{1}{2}\left(f(n \bmod q) + f(-n \bmod q)\right)$$

and

$$f_{odd}(n) = \frac{1}{2}\left(f(n \bmod q) - f(-n \bmod q)\right).$$

Then

(0.2) $$f = f_{even} + f_{odd}.$$

We say f is an even or odd function according as $f = f_{even}$ or $f = f_{odd}$.

$L(s,f)$ satisfies a ramified type functional equation due to Schnee[10]:

(0.3) $$L(1-s,f) = \left(\frac{q}{\pi}\right)^{s-1/2}\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}L(s,\hat{f}_{even}) + i\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}L(s,\hat{f}_{odd})\right)$$

(cf. Apostol[1], Funakura[5])

§ 1. Special values of $L(s,f)$ and values $L^{(k)}(1,f)$

From Formula (0.1) , taking into account that the term with $a=q$ is $\frac{1}{\sqrt{q}}\hat{f}(q)\zeta(s)$, we get

Theorem 1

$$L(s,f) = \frac{\hat{f}(q)}{\sqrt{q}}\zeta(s) + \frac{1}{\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)F(s,a/q),$$

in particular, for $k \in \mathbf{N}$,

$$\sqrt{q}\left(L(k,f) - \frac{\hat{f}(q)}{\sqrt{q}}\zeta(k)\right) = \frac{(2\pi i)^{k-1}}{k!}\sum_{a=1}^{q-1}\hat{f}(a)A_k\left(\frac{a}{q}\right) - \frac{(2\pi i)^k}{2k!}\sum_{a=1}^{q-1}\hat{f}(a)B_k\left(\frac{a}{q}\right),$$

where $B_k(x)$ is the k-th Bernoulli polynomial and $A_k(x)$ is essentially the Clausen function(cf. Kanemitsu [7]), and where for k=1and

(1.1) $$\sqrt{q}\hat{f}(q) = f(1) + \cdots + f(q) \neq 0,$$

the left-hand side is to be understood to mean

$$\lim_{s \to 1} \sqrt{q}\left(L(s,f) - \frac{\hat{f}(q)}{\sqrt{q}}\zeta(s) \right).$$

Of course, if $\sqrt{q}\hat{f}(q) = 0$, the formula gives the special values $L(k,f)$.

From now on, for q=1, the empty sum $\sum_{a=1}^{q-1}$ is taken to mean 0 so as to include the case of q=1, $L(s,f) = \zeta(s)$.

Definition We call f principal if $\hat{f}(q) \neq 0$ and non-principal otherwise.

Corollary 1 (Generalization of Proposition 4.1 of Yamamoto [11])

If f is non-principal, i.e. if $\hat{f}(q) \neq 0$, then for every $s \in \mathbf{C}$,

$$L(s,f) = \frac{1}{\sqrt{q}} \sum_{a=1}^{q-1} \hat{f}(a) F(s, a/q).$$

In particular, for $k \in \mathbf{N}$,

$$L(k,f) = \begin{cases} \dfrac{1}{\sqrt{q}} \dfrac{(2\pi i)^{k-1}}{k!} \displaystyle\sum_{a=1}^{q-1} \hat{f}(a) A_k\left(\frac{a}{q}\right), & \text{if } f \text{ and } k \text{ have the opposite parity.} \\ -\dfrac{1}{\sqrt{q}} \dfrac{(2\pi i)^{k}}{k!} \displaystyle\sum_{a=1}^{q-1} \hat{f}(a) B_k\left(\frac{a}{q}\right), & \text{if } f \text{ and } k \text{ have the same parity.} \end{cases}$$

Proof. The proof of this Corollary as well as other results pertaining to evenness and oddness rests on the correspondence : Evenness and oddness of f and *a fortiori*, of $L(s,f)$ corresponds to that of the basis of K_s, the Kubert vector space of dimension 2, Milnor [8]. In this case, according as k is even or odd, $A_k(a/q)$ is odd or even, and

$B_k(a/q)$ is even or odd, the result now follows from

$$L(k,f_{odd})=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{f(n)-f(-n)}{n^k}\right),\qquad L(k,f_{even})=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{f(n)+f(-n)}{n^k}\right).$$

Corollary 2 (Corollary 6 of Funakura [6])

(1) $$L(1,f_{odd})=\frac{\pi}{2q}\sum_{a=1}^{q-1}f(a)\cot\frac{\pi a}{q},$$

(2) $$L(1,f_{even})=-\frac{1}{\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)\log\sin\frac{\pi a}{q}-f(q)\log 2.$$

Proof. This follows immediately from Corollary 1, and the following formula which will also be used subsequently :

(1,2) $$\left(\sum_{a=1}^{q-1}\hat{f}(a)A_1(a/q)=\right)-\sum_{a=1}^{q-1}\hat{f}(a)\log 2\sin\frac{\pi a}{q}$$

$$=-\sqrt{q}f(q)+\hat{f}(q)-\sum_{a=1}^{q-1}\hat{f}(a)\log\sin\frac{\pi a}{q}$$

(where the last term can be expressed further as

$$-\frac{1}{2\sqrt{q}}\sum_{n=1}^{q}f(n)\sum_{a=1}^{q-1}e\left(-\frac{na}{q}\right)\log 2\sin\frac{\pi a}{q}\Bigg)$$

(1.3) $$\sum_{a=1}^{q-1}\hat{f}(a)\bar{B}_1(a/q)=\frac{i}{2\sqrt{q}}\sum_{r=1}^{q-1}f(r)\cot\frac{\pi r}{q}$$

Corollary 3 (Dirichlet class number formula in finite form)

If χ is a non-principal Dirichlet character mod q, then

(i) $$L(1,\chi_{odd})=\frac{\pi}{2q}\sum_{a=1}^{q-1}\chi(a)\cot\frac{\pi a}{q}$$

(ii) $$L(1,\chi_{even})=-\frac{1}{q}\sum_{a=1}^{q-1}\hat{\chi}(a)\log\sin\frac{\pi a}{q}\ \ (\chi \text{ not necessarily primitive})$$

$$=-\frac{G(\chi)}{q}\sum_{a=1}^{q-1}\overline{\chi}(a)\log\sin\frac{\pi a}{q}\ \ (\chi \text{ primitive}),$$

where $G(\chi)=\sum_{a \bmod q}\chi(a)\xi_a(1)$ is the normalized Gauss sum.

Theorem 2. At $s=1$, $L(s,f)$ has the Laurent or Taylor expansion according as f is principal or not :

$$L(s,f)=\frac{\gamma_{-1}(f)}{s-1}+\gamma_0(f)+\sum_{n=1}^{\infty}\frac{\gamma_n(f)}{n!}(s-1)^n,$$

where

$$\gamma_{-1}(f)=\frac{1}{\sqrt{q}}\hat{f}(q),$$

(1.4) $$\gamma_0(f)=\frac{1}{\sqrt{q}}\hat{f}(q)\gamma-\frac{1}{\sqrt{q}}\sum_{r=1}^{q-1}\hat{f}(r)\log 2\sin\frac{\pi r}{q}-f(q)\log 2+\frac{\pi}{2q}\sum_{r=1}^{q-1}f(r)\cot\frac{\pi r}{q}$$

(Theorem 5 of Funakura [6])

(1.5) $$\gamma_1(f)=\frac{1}{\sqrt{q}}\hat{f}(q)\gamma_1-\frac{1}{\sqrt{q}}(\gamma+\log 2\pi)\sum_{a=1}^{q-1}\hat{f}(a)\log 2\sin\frac{\pi a}{q}$$

$$+\frac{1}{2\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)\left(R\left(\frac{a}{q}\right)+R\left(1-\frac{a}{q}\right)\right)-\frac{\pi i}{\sqrt{q}}(\gamma+\log 2\pi)\sum_{a=1}^{q-1}\hat{f}(a)B_1\left(\frac{a}{q}\right)$$

$$-\frac{\pi i}{2\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)\left(\log\Gamma\left(\frac{a}{q}\right)-\log\Gamma\left(1-\frac{a}{q}\right)\right),$$

(1.6) $$\gamma_2(f)=\frac{1}{\sqrt{q}}\hat{f}(q)\gamma_2+\frac{1}{3\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)\left(R_3\left(\frac{a}{q}\right)+R_3\left(1-\frac{a}{q}\right)\right)$$

$$+\frac{1}{\sqrt{q}}(\gamma+\log 2\pi)\sum_{a=1}^{q-1}\hat{f}(a)\left(R\left(\frac{a}{q}\right)-R\left(1-\frac{a}{q}\right)\right)$$

$$-\frac{1}{\sqrt{q}}\left(\log^2 2\pi+\zeta(2)+\gamma^2+2\gamma\log 2\pi-\frac{\pi^2}{4}\right)\sum_{a=1}^{q-1}\hat{f}(a)\log 2\sin\frac{\pi a}{q}$$

$$-\frac{\pi i}{2\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)\left(R\left(\frac{a}{q}\right)-R\left(1-\frac{a}{q}\right)\right)-\frac{\pi i}{\sqrt{q}}(\gamma+\log 2\pi)\sum_{a=1}^{q-1}\hat{f}(a)\left(\log\Gamma\left(\frac{a}{q}\right)-\log\Gamma\left(1-\frac{a}{q}\right)\right)$$

$$-\frac{\pi i}{\sqrt{q}}\left(\log^2 2\pi+\zeta(2)+\gamma^2+2\gamma\log 2\pi-\frac{\pi^2}{12}\right)\sum_{a=1}^{q-1}\hat{f}(a)B_1\left(\frac{a}{q}\right),$$

etc., where γ_i are generalized Euler constants ($\gamma_0=\gamma$, the Euler constant ; for more details about generalized Euler constants, cf. § 2-1). In particular, if f is non-principal, then $L^{(k)}(1,f)=\gamma_{k+1}(f)$, so that above formulas give closed form evaluation of the value of the derivatives at s=1.

Further simplication of (1.4) using (1.2) leads to formulas in Corollary 2 of Theorem 1. Similarly, Formulas (1.3), (1.4) can be expressed without Fourier transforms of f.

Proof. The proof easily follows from the formula in Theorem 1. Namely, since $\zeta(s)$ has the well-known Laurent expansion

$$\zeta(s)=\frac{1}{s-1}+\gamma+\sum_{n=1}^{\infty}\gamma_n(s-1)^n \tag{1.7}$$

with generalized Euler constants and $F(s,a/q)$ $(1\le a<q)$ has a Taylor expansion,

$$F\left(s,\frac{a}{q}\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial s^n}F\left(1,\frac{a}{q}\right),$$

as an integral function of s, it follows that

$$\gamma_{-1}(f)=\frac{1}{\sqrt{q}}\hat{f}(q),$$

and

$$\gamma_n(f)=\frac{\hat{f}(q)}{\sqrt{q}}\gamma_n+\frac{1}{\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)\frac{\partial^n}{\partial s^n}F\left(1,\frac{a}{q}\right),\ n\in\mathbf{N}.$$

The closed form for $\frac{\partial^n}{\partial s^n}F\left(1,\frac{a}{q}\right)$ (n=0,1,2) is given by Kanemitsu [7].

Corollary 1.

(i) $$L'(1,f_{even}) = -\frac{1}{\sqrt{q}}(\gamma+\log 2\pi)\sum_{a=1}^{q-1}\hat{f}(a)\log 2\sin\frac{\pi a}{q} + \frac{1}{\sqrt{q}}\sum_{a=1}^{q}\hat{f}(a)R\left(\frac{a}{q}\right)$$

(ii) $$L'(1,f_{odd}) = -\frac{\pi i}{\sqrt{q}}(\gamma+\log 2\pi)\sum_{a=1}^{q-1}\hat{f}(a)\overline{B}_1\left(\frac{a}{q}\right) - \frac{1}{\sqrt{q}}\sum_{a=1}^{q}\hat{f}(a)\log\Gamma\left(\frac{a}{q}\right)$$

Corollary 2.

(i) $$L''(1,f_{even}) = -\frac{2}{3\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)R_3\left(\frac{a}{q}\right) - \frac{2}{\sqrt{q}}(\log 2\pi+\gamma)\sum_{a=1}^{q-1}\hat{f}(a)R\left(\frac{a}{q}\right)$$

$$+\frac{1}{\sqrt{q}}\left(\log^2 2\pi + \zeta(2) + \gamma^2 + 2\gamma\log 2\pi - \frac{\pi}{4}\right)\sum_{a=1}^{q-1}\hat{f}(a)\log 2\sin\frac{\pi a}{q}$$

(ii) $$L''(1,f_{odd}) = \frac{\pi i}{\sqrt{q}}\sum_{a=1}^{q-1}\hat{f}(a)R\left(\frac{a}{q}\right) + \frac{2\pi i}{\sqrt{q}}(\gamma+\log 2\pi)\sum_{a=1}^{q-1}\hat{f}(a)\log\Gamma\left(\frac{a}{q}\right)$$

$$+\frac{\pi i}{\sqrt{q}}\left(\log^2 2\pi + \zeta(2) + \gamma^2 + 2\gamma\log 2\pi - \frac{\pi^2}{12}\right)\sum_{a=1}^{q-1}\hat{f}(a)\overline{B}_1\left(\frac{a}{q}\right).$$

Both of these Corollaries immediately follow from Theorem 2 by separating even and odd parts (i.e. summing the term with a and q-a in pairs).

Remark. Corollary 1 with $f=\chi$ (non-principal Dirichlet character) and

$$\hat{\chi} = \frac{\chi(-1)G(\chi)}{\sqrt{q}}\overline{\chi}$$

gives Chowla-Selberg formula and Deninger's formula. Similarly Corollary 2, (i), (ii) give generalizations of formulas of Kanemitsu [7].

Further derivative $L^{(k)}(1,f), k \geq 3$ can be calculated in principle by using the function $R_{k+1}(x)$, on the basis of

$$L^{(k)}(s,f) = \frac{1}{\sqrt{q}} \sum_{a=1}^{q-1} \hat{f}(a) F^{(k)}\left(s, \frac{a}{q}\right).$$

§ 2. Products of L-functions

§ 2-1. A generalization of the Piltz divisor problem.

We consider the k-fold product of the L-function with a principal f:

$$Z(s) = Z_k(s) = L(s,f)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s},$$

where we write $d_k(n) = d_{k,f}(n) = \sum_{d_1 \cdots d_k = n} f(d_1) \cdots f(d_k)$, the k-fold divisor-like sum (so that for $f \equiv 1$, $d_{k,f}(n) = d_k(n)$, the ordinary divisor function).

The summatory function $D_k(x) = D_{k,f}(x) = \sum_{n \leq x} d_k(n)$ admits the asymptotic formula

$$D_k(x) = x P_k(\log x) + \Delta_k(x),$$

where the main term $x P_k(\log x) = x\left(a_{k-1}^{(k)} (\log x)^{k-1} + \cdots + a_1^{(k)} \log x + a_0^{(k)}\right)$ is the residue of $Z(s) x^s / s$ at x=1 and $\Delta_k(x)$ is the associated error term. Under the assumption

$$f(n) << n^{\varepsilon}, \ ^{\forall}\varepsilon > 0,$$

it is easily proved by elementary means that

$$\Delta_k(x) << x^{1-1/k}$$

(the estimate can be improved by using more subtle techniques).

Lavrik [8] has generally been considered as the first who expressed the coefficients $a_i^{(k)}$ in terms of generalized Euler constants γ_r defined by

$$(2.2) \qquad \gamma_r = \frac{(-1)^r}{r!} \lim_{x\to\infty}\left(\sum_{n\le x}\frac{\log^r n}{n} - \frac{1}{r+1}\log^{r+1} x\right), \quad r = 0,1,2,\cdots,$$

but preceding Lavrik by more than 20 tears, Eda [6] had made the same thing, except for estimation and representation of γ_r . Eda also referred to a more general problem for the product of k L-functions, but he gave a representation of $L(1,\chi)$ only, so that it can hardly be said the determination of $a_i^{(k)}$ for this problem was done. Finite form expressions for $L^{(k)}(1,\chi)$ have been given by Deninger [3] for k=0,1 and by Kanemitsu [7] for $0 \le k \le 2$ (similar formulas hold for general k).

The generalized Euler constants γ_r appear as the Laurent coefficients of $\zeta(s)$ at s=1 (it is more customary to define γ_r by (2.2) without the factor $(-1)^r/r!$, but to conform with the notation of Lavrik [8]and Sitaramachandrarao [12] we adopted the above notation. Anyway, these were already known to Stieltjes and have been considered by several authors including Briggs, Chowla, Hardy, Kluyver et al (cf. Ref. of Berndt's paper [2]).

In that paper, Berndt obtained the Laurent coefficients of the Hurwitz zeta-function, generalizing former results due to Wilton, but he does not mention the paper of Daojoy [5] whose results were the basis of Eda's investigation. However, Berndt had better estimates, which have been subsequently improved by Balakrishnan [1]. For sharp estimates for γ_r's themselves, see the papers of Matsuoka [11].

The expression of Eda-Lavrik for $a_i^{(k)}$ led Lavrik, Israilov and Ëdgorov [9] to express the values of integral

$$I_k = I_k(f) = \int_1^{\infty} \frac{\Delta_k(u)}{u^2} du$$

in terms of γ_i 's. More recently, Sitaramachandrarao [12] has simplified the arguments of Lavrik [8] and Lavrik et al [9], giving lucid relation among those constants.

We now follow Sitaramachandrarao's argument (still simplifying it) to prove his formula (1.2), (1.3), (1.4).

By partial summation

$$\sum_{n \le x} d_k(n) n^{-s} = x^{-s} D_k(x) + s \int_1^x \frac{u \sum_{i=0}^{k-1} a_i^{(k)} (\log u)^i + \Delta_k(u)}{u^{s+1}} du .$$

Letting $x \to \infty$ for $\sigma > 1$, we get

$$Z(s) = s\left(\sum_{i=0}^{k-1} a_i^{(k)} i!(s-1)^{-i-1} + \int_1^{\infty} \frac{\Delta_k(u)}{u^{s+1}} \right)$$

or

(2.3) $$f_k(s) := \int_1^{\infty} \frac{\Delta_k(u)}{u^{s+1}} du = \frac{Z(s)}{s} - \sum_{i=1}^{k-1} a_i^{(k)} i!(s-1)^{-i-1} .$$

By (2.1), the integral defining $f_k(s)$ is absolutely and uniformly convergent in the wide sense in the region $\sigma > 1 - 1/k$, whence $f_k(s)$ is analytic there. In particular, the right-hand side is also analytic at s=1, to the effect that the principal part of $Z(s)/s$ around s=1 is exactly the second summand (with minus sign omitted) of (2.3).

For comparison's sake we now deduce the Laurent expansion of $Z(s)/s$ $= L(s,f)^k/s$ from that of $L(s,f)$. Writting $\tilde{\gamma}_n = \tilde{\gamma}_n(f) = \gamma_{n-1}(f)$, we have from Theorem 2

$$\frac{1}{s}Z(s) = \left(\sum_{n=-1}^{\infty}\gamma_n(f)(s-1)^{n+1}\right)^k (s-1)^{-k}(s-1+1)^{-1}$$

$$= \left(\sum_{n=0}^{\infty}\tilde{\gamma}_n(f)(s-1)^n\right)^k \sum_{n=0}^{\infty}(-1)^n(s-1)^n(s-1)^{-k}$$

$$= \sum_{n=0}^{\infty}\beta_n^{(k)}(s-1)^n(s-1)^{-k},$$

say, where

$$\beta_n^{(k)} = \beta_n^{(k)}(f) = \sum_{\substack{i+i_1+\cdots+i_k=n \\ i_l\ge 0}}(-1)^i\tilde{\gamma}_{i_1}\cdots\tilde{\gamma}_{i_k} = \sum_{r=0}^{n}(-1)^{n-r}\sum_{\substack{i_1+\cdots+i_k=r \\ i_l\ge 0}}\tilde{\gamma}_{i_1}\cdots\tilde{\gamma}_{i_k},$$

so that

$$\beta_n^{(k)} = (-1)^n\gamma_{-1}(f)^k + \sum_{r=1}^{n}(-1)^{n-r}\sum_{\substack{i_1+\cdots+i_k=r \\ i_l\ge 0}}\gamma_{i_1-1}(f)\cdots\gamma_{i_k-1}(f).$$

We separate the innermost sum according to how many (, say s) of i_l's are non-zero, and obtain

$$(2.3)\quad \beta_n^{(k)} = (-1)^n\left(\gamma_{-1}(f)^k + \sum_{r=1}^{n}(-1)^r\sum_{s=1}^{r}\binom{k}{s}\gamma_{-1}(f)^{k-s}\sum_{\substack{i_1+\cdots+i_k=r-s \\ i_1,\cdots,i_s\ge 1}}\gamma_{i_1}(f)\cdots\gamma_{i_s}(f)\right).$$

Since the principal part of the Laurent expansion of $Z(s)/s$ is

$$\sum_{i=0}^{k-1}\beta_{k-1-i}^{(n)}(s-1)^{-i-1},$$

it follows that

$$a_i^{(k)} = \frac{\beta_{k-1-i}^{(k)}}{i!},\qquad i = 0,\cdots,k-1,$$

which is (1.3) of [12], Theorem 1 of [8], Theorem 2 of [6].

Similarly, the constant term is $\beta_k^{(k)}$, which is equal to $f_k(1) = I_k$, giving Formula (1.4) of [12] :

$$I_k = \int_1^\infty \frac{\Delta_k(u)}{u^2} du = f_k(1) = \beta_k^{(k)}.$$

Finally, to prove Formula (1.2), we need a recurrence between $\beta_n^{(k+1)}$ and $\beta_n^{(k)}$, which is obtained as follows. We have

$$\frac{1}{s} Z_{k+1}(s) = \sum_{n=0}^{\infty} \beta_n^{(k+1)} (s-1)^n (s-1)^{-k-1},$$

or

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n^{(k+1)} (s-1)^n &= \frac{1}{s} \big((s-1) L(s,f)\big)^{k+1} \\ &= \frac{1}{s} Z_k(s) (s-1)^k (s-1) L(s,f) \\ &= \sum_{n=0}^{\infty} \beta_n^{(k)} (s-1)^n \sum_{n=0}^{\infty} \tilde{\gamma}_n (s-1)^n, \end{aligned}$$

whence

$$\beta_n^{(k+1)} = \sum_{i=0}^{n} \tilde{\gamma}_i \beta_{n-i}^{(k)} = \beta_n^{(k)} + \sum_{i=0}^{n-1} \gamma_i(f) \beta_{n-i-1}^{(k)}.$$

In particular,

$$\begin{aligned} I_k = \beta_k^{(k)} &= \beta_k^{(k+1)} - \sum_{i=0}^{k-1} \gamma_i(f) \beta_{n-i-1}^{(k)} \\ &= a_0^{(k+1)} - \sum_{i=0}^{k-1} i!\, \gamma_i(f) a_i^{(k)}, \end{aligned}$$

which is the theorem of Lavrik, Israilov and Ëdgorov [9].

§ 2-2. A generalization of a theorem of Müller-Carlitz-Ayoub-Chowla-Redmond-Berndt

Consider the convolution $r = f * g$ with f,g both even and period q,q', respectively (aiming at an application to the case of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d>0$: $\zeta_K(s)=\zeta(s)L\left(s,\left(\frac{\cdot}{d}\right)\right)$, where both $f\equiv 1$, $g=\left(\frac{\cdot}{d}\right)$, the Kronecker symbol, are even), and the associated L-series

$$Z(s) = L(s,f)L(s,g) = \sum_{n=1}^{\infty}\frac{r(n)}{n^s},\ \sigma > 1.$$

The authors in the title (except for Berndt, who considers far more general class of Dirichlet series) were interested in the asymptotic formula for the logarithmic Riesz sum (of order 1)

$$S(x) = S(x;1) = \sum_{n\le x} r(n)\log\frac{x}{n} = \frac{1}{2\pi i}\int_{(c)} Z(s)\frac{x^s}{s^2}ds, \quad \text{with } c>1.$$

We note the following estimate for $L(s,f)$:

$$L(s,f) << \begin{cases} 1, & \sigma>1, \\ q^{\frac{1-\sigma}{2}+\varepsilon}(q|t|)^{\frac{1-\sigma}{2}}\log|t|, & 0\le\sigma\le 1, \\ (q|t|)^{\frac{1}{2}-\sigma}\log|t|, & \sigma<0, \end{cases}$$

which follows from absolute convergence, the corresponding order estimate for $\zeta(s,\alpha)$ and (0.0), and functional equation (0.3), respectively.

Hence, by Cauchy's theorem, we can shift the line of integration to $1-c$, encountering two poles at $s=0,1$. The point 1 is a pole only when at least one of f, g is principal. Hence,

$$\text{(2.4)} \qquad S(x) = \left(\operatorname*{Res}_{s=1} + \operatorname*{Res}_{s=0}\right) Z(s)\frac{x^s}{s^2} + \frac{1}{2\pi i}\int_{(1-c)} Z(s)\frac{x^s}{s^2}ds$$

$$= M(x)+E(x),$$

say. Redmond [8] made an explicit determination of the coefficients in $M(x)$ with f,g Dirichlet characters, which we will first generalize.

In the neighborhood of $s=0$,

$$Z(s)\frac{x^s}{s^2} = Z(0)s^{-2} + (C_3 \log x + C_4)s^{-1} + \cdots$$

with

$$C_3 = C_3(qq') = Z(0) = L(0,f)L(0,g),$$

$$C_4 = C_4(qq') = Z'(0) = L'(0,f)L(0,g) + L(0,f)L'(0,g),$$

which can be explicitly expressed in view of

$$L(0,f) = \sum_{a=1}^{q} f(a)\zeta\left(0,\frac{a}{q}\right) = \sum_{a=1}^{q} f(a)\left(\frac{1}{2} - \frac{a}{q}\right)$$

and

$$L'(0,f) = \sum_{a=1}^{q} f(a)\zeta'\left(0,\frac{a}{q}\right) = \sum_{a=1}^{q} f(a)\left(\log\Gamma\left(\frac{a}{q}\right) - \frac{1}{2}\log 2\pi\right).$$

In the same way as Redmond [8] we get

$$\operatorname*{Res}_{s=1} Z(s)\frac{x^s}{s^2} = C_1 x\log x + C_2 x,$$

where

$$C_1 = C_1(qq') = \begin{cases} \gamma_{-1}(f)\gamma_{-1}(g), & \text{if both } f,g \text{ are principal,} \\ 0, & \text{otherwise,} \end{cases}$$

$$C_2 = C_2(qq') = \begin{cases} -2\gamma_{-1}(f)\gamma_{-1}(g) + \gamma_{-1}(f)\gamma_0(g) + \gamma_0(f)\gamma_{-1}(g), & \\ & \text{if at least one of } f,g \text{ is principal,} \\ 0, & \text{if neither of } f,g \text{ is principal.} \end{cases}$$

Hence

$$M(x) = C_1 x \log x + C_2 x + C_3 \log x + C_4. \tag{2.5}$$

We make a parenthetical note on $\gamma_0(f)$ when $f = \chi_0$, a principal character mod q: On the one hand

$$\gamma_0(\chi_0) = \frac{\varphi(q)}{q}\gamma - \frac{1}{\sqrt{q}}\sum_{r=1}^{q-1}\hat{f}(r)\log 2\sin\frac{\pi r}{q} + \frac{\pi}{2q}\sum_{r=1}^{q-1} f(r)\cot\frac{\pi r}{q}$$

by Theorem 2, and on the other hand, by Redmond's Lemma 4,

$$\gamma_0(\chi_0) = \frac{\varphi(q)}{q}\left\{\sum_{p|q}\frac{\log p}{p-1} + \gamma\right\},$$

whence

$$\varphi(q)\sum_{p|q}\frac{\log p}{p-1} = -\sqrt{q}\sum_{r=1}^{q-1}\hat{f}(r)\log 2\sin\frac{\pi r}{q} + \frac{\pi}{2}\sum_{\substack{r=1\\(r,q)=1}}^{q-1} f(r)\cot\frac{\pi r}{q}.$$

Now in the integral E(x), make the change of variable $s \leftrightarrow 1-s$ and use the functional equation (0.3) to obtain

$$E(x) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{1-s}{2}\right)}\hat{Z}(s)\left(\frac{q}{\pi}\right)^{s-1/2}\left(\frac{q'}{\pi}\right)^{s-1/2}\frac{x^{1-s}}{(1-s)^2}\,ds \tag{2.6}$$

$$= \frac{\sqrt{qq'}}{\pi}\sum_{n=1}^{\infty}\frac{\hat{f}*\hat{g}(n)}{n}\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma^2\left(\frac{s}{2}\right)\left(\frac{\pi^2}{qq'}nx\right)^{1-s}}{\Gamma^2\left(\frac{1-s}{2}\right)(1-s)^2}\,ds.$$

Thus we are led to evaluate the integral

$$I(x)=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma^2\left(\frac{s}{2}\right)x^{1-s}}{\Gamma^2\left(\frac{1-s}{2}\right)(1-s)^2}ds$$

for $x>0$ and $0<c<3/2$ (to secure absolute convergence).

We note that without loss of generality we may assume that $0<c<1/2$, in which case

$$x\frac{d}{dx}I(x)=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma^2\left(\frac{s}{2}\right)x^{1-s}}{\Gamma^2\left(\frac{1-s}{2}\right)(1-s)}ds$$

is exactly the integral $I_1(x)$ in the following (cf. [5], Lemma 7.1).

Lemma. Let $x>0$, $k\in \mathbf{N}$ and $o<c<k/2$. Then

$$I_k(x):=\int_{c-i\infty}^{c+i\infty}\frac{\Gamma^2\left(\frac{s}{2}\right)x^{k-s}}{\Gamma^2\left(\frac{1-s}{2}\right)(1-s)\cdots(k-s)}ds$$

$$=-2^{1-k}x^{k/2}F_k(4\sqrt{x})-4^{1-k}\pi^{-1}\begin{cases}\displaystyle\sum_{0\le 2m\le k-c}(4x)^{2m}\frac{\Gamma(k-2m)}{\Gamma(2m+1)}, & k\equiv 0\,(2),\\ \displaystyle\sum_{1\le 2m-1\le k-c}(4x)^{2m-1}\frac{\Gamma(k-2m+1)}{\Gamma(2m)}, & k\equiv 1\,(2),\end{cases}$$

where, following Voronoĭ, we put

$$F_\nu(x)=Y_\nu(x)+(-1)^{\nu-1}\frac{2}{\pi}K_\nu(x),$$

and $Y_\nu(x), K_\nu(x)$ are ordinary and modified Bessel functions usually so denoted.

Thus

$$\frac{d}{dx}I(x)=-\frac{1}{\sqrt{x}}F_1(4\sqrt{x}),$$

or

$$I(x) = -\int_{0+}^{x} \frac{1}{\sqrt{x}} F_1(4\sqrt{x})dx$$

$$= -\frac{1}{2}\int_{0+}^{4\sqrt{x}} F_1(y)dy$$

$$= -\frac{1}{2}\int_{0+}^{4\sqrt{x}} \left(Y_1(y) + \frac{2}{\pi} K_1(y) \right) dy.$$

Now noting

$$K_0'(y) = -K_1(y), \quad [10], \text{ p.79, (7)},$$

$$Y_0'(y) = -Y_1(y)\ , \quad [10], \text{ p.66, (14)},$$

and

$$Y_1(0) + \frac{2}{\pi} K_1(0) = 0,$$

we conclude that

$$I(x) = \frac{1}{2}\left(Y_1(4\sqrt{x}) + \frac{2}{\pi} K_1(4\sqrt{x}) \right). \tag{2.7}$$

Now from (2.4)-(2.7) we have

Theorem 3. If f, g are both even, then

$$\sum_{n \le x} r(n) \log\frac{x}{n} = C_1 x \log x + C_2 x + C_3 \log x + C_4$$

$$+ \frac{\sqrt{qq'}}{2\pi} \sum_{n=1}^{\infty} \frac{\hat{f} * \hat{g}(n)}{n} \left(Y_1\left(4\frac{\pi}{\sqrt{qq'}}\sqrt{nx} \right) + \frac{2}{\pi} K_1\left(4\frac{\pi}{\sqrt{qq'}}\sqrt{nx} \right) \right).$$

The study of other cases will be given elsewhere.

We only note the special case of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d}), d > 0$, in which case $f \equiv 1, g = \left(\frac{-d}{\cdot}\right)$, the Kronecker character. The Dedekind zeta function $\zeta_K(s) = \sum_{n=1}^{\infty} \frac{F(n)}{n^s}$ satisfies the functional equation of Hecke type and the following has been given by Berndt [2].

$$\text{(2.7)} \quad \sum_{n \le x} F(n) \log \frac{x}{n} = \lambda h x + \zeta_K(0) \log x + \zeta_K'(0) - \frac{\sqrt{|d|}}{2\pi} \sum_{n=1}^{\infty} \frac{F(n)}{n} J_0\left(4\pi \sqrt{\frac{nx}{|d|}}\right),$$

which reduces, in the case $\mathbb{Q}(\sqrt{-4})$, to

$$\text{(2.8)} \qquad \sum_{n \le x} r_2(n) \log \frac{x}{n} = \pi x - \log x + \zeta_2'(0) - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r_2(n)}{n} J_0\left(2\pi \sqrt{nx}\right),$$

where

$$r_2(n) = \sum_{a^2 + b^2 = n} 1 = 4F(n) = 4 \sum_{N\mathfrak{a} = n} 1, \text{ and } \zeta_2(s) = \sum_{n=1}^{\infty} \frac{r_2(n)}{n^s},$$

Comparing this with the formula obtained by Müller [6], Carlitz [3] or Redmond [8] :

$$\text{(2.9)} \qquad \sum_{n \le x} r(n) \log \frac{x}{n} = \pi x - \log x - \log \frac{\Gamma^4(1/4)}{4\pi} + O(x^{-1/4}),$$

(on noting that the first term of the asymptotic formula for J_0 gives the same error estimate) we get

$$\zeta_{\mathbb{Q}(\sqrt{-4})}'(0) = \zeta_2'(0) = -\log \frac{\Gamma^4(1/4)}{4\pi}.$$

Incidentally, Nowak [7] has also proved (2.8) with different evaluation of the constant term, whence we deduce in particular the following strange evaluation :

$$\text{(2.10)} \qquad \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r_2(n)}{n} J_0(2\pi \sqrt{nu}) = \pi u - \log u - \log \frac{\Gamma^4(1/4)}{4\pi}, \text{ for } 0<u<1.$$

On § 2-3 and § 3 only a brief mention is made of some possible directions of research and details will be given elsewhere.

Some of the most natural products of L-functions are

(1) $\zeta(s)L\left(s,\left(\frac{d}{\cdot}\right)\right)=\zeta_{\mathbb{Q}(\sqrt{d})}$, the Dedekind zeta-function of the quadratic field $\mathbb{Q}(\sqrt{d})$.

(2) $L_{d_1}(s)L_{d_2}(s)=L_K(s,\chi)$, the L-series with a genus character χ for the quadratic field $K=\mathbb{Q}(\sqrt{d})$, where d_1, d_2 arise from the decomposition $d=d_1d_2$.

(3) $\zeta(s)\prod_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} L(s,\chi)=\zeta_K(s)$, the Dedekind zeta-function for the maximal real subfield $K=\mathbb{Q}(\zeta+\zeta^{-1})$ of the cyclotomic field $\mathbb{Q}(\zeta_p)$, where ζ is the p-th root of unity $\zeta=\exp(2\pi i/p)$, p a prime.

(4) $L(s,\chi_1)\cdots L(s,\chi_k)=Z_k(s)$, the product of k L-functions, not necessarily primitive, considered by Gel'fond, Landau, Suetuna, ...

(1) - (3) and generalization thereof are of arithmetic flavor and usually lead to the Chowla-Selberg type formulas, while for the product of type (4) one can build up an analogous theory to the well-known case $\zeta(s)^k$, i.e. approximate functional equation, mean value theorems, etc.

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